

Adaptive Observers as Nonlinear Internal Models*

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Abstract

This paper shows how the theory of nonlinear adaptive observers can be effectively used in the design of internal models for nonlinear output regulation. The theory substantially enhances the existing results in the context of *adaptive* output regulation, by allowing for not necessarily stable zero dynamics of the controlled plant and by weakening the standard assumption of having the steady state control input generated by a linear system.

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1 Introduction

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances, sometimes known as the *servomechanism problem*, is continuing to attract a good deal of interest in control theory. In the last decade or so, most of the efforts have been addressed toward the design of controllers which solve this problem in the case of plants modelled by nonlinear differential equations. Viewed as a nonlinear design problem, some of the original features (such as, for instance, the conservation of the desired steady-state features in spite of plant parameter variations, otherwise known as “robustness” property, and the necessity of an “internal model” in any robust regulator) tend to lose their specific connotation. Rather, they merge with other relevant issues in feedback design for nonlinear systems, notably the guarantee of convergence

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(for certain state variables) or boundedness (for other state variables) once a fixed set of initial data is given. After all, a problem of steering certain variables to a desired target value in the presence of exogenous stimuli generated by an autonomous “exosystem”, in a nonlinear context, can be viewed as a problem of adaptive control. Classically, adaptation is sought with respect to uncertain but constant parameters (which, of course, can be seen as exogenous inputs generated by a “trivial” autonomous exosystem) but if the parameters in question are obeying some fixed differential equation, it would not be inappropriate to continue to call a problem of this kind a (generalized) problem of adaptive control. It is because of this observation that an increasing use of methods and techniques from (nonlinear) adaptive control should be expected, so long as newer results in this research area will be generated.

A specific feature of the problem in question is that, to achieve the desired steady-state features (perfect tracking), the controller should be able to generate a family of special inputs (those which, in fact, secure perfect tracking). If the external stimuli are constant (as in the case of uncertain constant plant parameters), this generator is simply provided by a bank of integrators (in the case of adaptive control, the state of each of such integrator is a *parameter estimate*). These integrators are to be “controlled” by appropriate feedback laws, so as to achieve the desired convergence and/or boundedness properties (in the case of adaptive control, this is the design of the “adaptation laws”). In a general servomechanism problem, the setting is very much the same: a model which generates all inputs needed to obtain perfect tracking is first found (the “internal model”) and then some (generalized) stabilization law is superimposed to complete the design. In this respect, the design of a controller that solves the servomechanism problem is split in two parts: the design of an internal model and the design of a stabilizer. It must be observed, though, that the design of the former must be done with an eye to our ability to find the latter. This is a fact that was well understood in adaptive control. The addition of appropriate dynamics (notably the so-called “filtered transformations”) are not strictly speaking needed to provide the dynamics of the unknown parameters (a trivial dynamics in that case), but rather are introduced to address the issue of stability.

This being said, it is natural to expect an increasing interaction between the work on nonlinear adaptive control, nonlinear stabilization and nonlinear servomechanism theory. This interaction has already manifested itself in a number of recent contributions (such as [20], [3], [14], [8], [5], [19], whose results cannot be reviewed here for obvious reasons) and will manifest more in the contributions to come. In this paper, we wish to propose a method for design of an internal model which is based on some classical results in adaptive control: the design of adaptive observers for nonlinear systems which are linearizable by output injection. This method enables us to solve a servomechanism problem when the inputs needed to achieve perfect tracking can be seen as generated by a nonlinear system, linearizable by output injection, with possibly unknown coefficients. Allowing for unknown coefficients in this model automatically settles the issue of uncertain plant parameters (the classical “robustness” issue) as well the issue of parameter uncertainties in the exosystem (an outstanding design problem first addressed and solved, for a special class of systems, in [20]). Although the inspiration for the design is taken from an existing result in adaptive

control, the application to the specific context of servomechanism theory looks pretty new and worth being pursued.

2 Class of systems and main assumptions

2.1 Preliminaries

The purpose of this paper is to show how the theory of nonlinear adaptive observed can be effectively used in the design of adaptive output regulators for nonlinear systems. To motivate why and how adaptive observers play an important role in this design problem, it suffices to address the simplified case in which the controlled plant has relative degree 1 between the *control input* and the *regulated output*. This is what is done here, for reason of space. The extension of the design methodology to system having higher relative degree can be found in the more extended paper [9], along with some additional technical details.

Consider a system modelled by equations of the form

$$\begin{aligned}\dot{z} &= f_0(\varrho, w, z) + f_1(\varrho, w, z, e)e \\ \dot{e} &= q(\varrho, w, z, e) + u,\end{aligned}\tag{1}$$

with state $(z, e) \in \mathbb{R}^n \times \mathbb{R}$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, in which the exogenous inputs $\varrho \in \mathbb{R}^p$ and $w \in \mathbb{R}^s$ are generated by an exosystem modelled by equations of the form

$$\begin{aligned}\dot{\varrho} &= 0 \\ \dot{w} &= s(\varrho, w).\end{aligned}\tag{2}$$

In this model, $\varrho \in \mathbb{R}^p$ is a vector of constant uncertain parameters, the aggregate of a finite set of uncertain parameters affecting the controlled plant and another, possibly different, set of uncertain parameters affecting the generator of the exogenous input w . Note that system (1) has relative degree 1 between control input u and regulated output e .

The functions $f_0(\cdot)$, $f_1(\cdot)$, $q(\cdot)$, $s(\cdot)$ in (1) and (2) are assumed to be at least continuously differentiable. The initial conditions of (1) range on a set $Z \times E$, in which Z is a fixed *compact* subset of \mathbb{R}^n and $E = \{e \in \mathbb{R} : |e| \leq c\}$, with c a fixed number. The initial conditions of the exosystem (2) range on set $P \times W$ in which P and W are *compact* subsets of \mathbb{R}^p and, respectively, \mathbb{R}^s . In this framework the problem of output regulation is to design an output feedback regulator of the form

$$\begin{aligned}\dot{\zeta} &= \varphi(\zeta, e) \\ u &= \gamma(\zeta, e)\end{aligned}$$

such that *for all initial conditions* $(\varrho(0), w(0)) \in P \times W$ and $(z(0), e(0)) \in Z \times E$ *the trajectories of the closed-loop system are bounded and* $\lim_{t \rightarrow \infty} e(t) = 0$.

We retain in this paper some ideas introduced in [3], to which – to avoid duplications – the reader is referred. Among the concepts introduced and/or summarized in that paper, the notion of *omega limit set* $\omega(\mathbf{S})$ of a set \mathbf{S} plays a major role. This concept is a deep generalization of the classical concept, due to Birkhoff, of omega limit set of a point and

provides a rigorous definition of steady-state response in a nonlinear system (see [3] for details).

Remark. The regulated variable e of (1) may coincide with the physical “controlled” output of a given plant, or may as well represents a *tracking error*, namely the difference between a physical “controlled” output and its “reference” behavior. Thus, the problem under consideration includes problems of tracking as well as problems of disturbance attenuation. \triangleleft

2.2 Basic hypotheses

Augmenting (1) with (2) yields a system which, viewing u as input and e as output, has relative degree 1. The associated “augmented” zero dynamics, which is forced by the control

$$c(\varrho, w, z) = -q(\varrho, w, z, 0), \quad (3)$$

is given by

$$\begin{aligned} \dot{\varrho} &= 0 \\ \dot{w} &= s(\varrho, w) \\ \dot{z} &= f_0(\varrho, w, z). \end{aligned} \quad (4)$$

Occasionally, throughout the paper, we will rewrite the latter as

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}), \quad (5)$$

where $\mathbf{z} = \text{col}(\varrho, w, z)$. Accordingly, we set $\mathbf{Z} = P \times W \times Z$ and replace $c(\varrho, w, z)$ by $c(\mathbf{z})$ in (3).

In what follows, we retain three of the basic assumptions that were introduced in [3] and express certain properties of the augmented zero dynamics (4). The assumptions in question are the following ones:

Assumption (i): the set $P \times W$ is a differential submanifold (with boundary) of $\mathbb{R}^p \times \mathbb{R}^s$, invariant for (2). \triangleleft

Assumption (ii): there exists a compact subset \mathcal{Z} of $P \times W \times \mathbb{R}^n$ which contains the positive orbit of the set \mathbf{Z} under the flow of (5), and $\omega(\mathbf{Z})$ is a differential submanifold (with boundary) of $P \times W \times \mathbb{R}^n$. Moreover there exists a number $d_1 > 0$ such that

$$\mathbf{z} \in P \times W \times \mathbb{R}^n, \quad \text{dist}(\mathbf{z}, \omega(\mathbf{Z})) \leq d_1 \quad \Rightarrow \quad \mathbf{z} \in \mathbf{Z}. \quad \triangleleft$$

Remark. Since the positive orbit of the set \mathbf{Z} under the flow of (5) is bounded by hypothesis, the set $\omega(\mathbf{Z})$ is a nonempty, compact and invariant subset of $P \times W \times \mathbb{R}^n$ which uniformly attracts all trajectories of (5) with initial conditions in \mathbf{Z} . It can also be shown (as in [3]) that for every $(\varrho, w) \in P \times W$ there is $z \in \mathbb{R}^n$ such that $(\varrho, w, z) \in \omega(\mathbf{Z})$. \triangleleft

In what follows, for convenience, the set $\omega(\mathbf{Z})$ will be simply denoted as \mathcal{A}_0 . The last condition in assumption (ii) implies that \mathcal{A}_0 is stable in the sense of Lyapunov. The next hypothesis is that the set \mathcal{A}_0 is locally exponentially attractive.

Assumption (iii): There exist $M \geq 1$, $a > 0$ and $d_2 \leq d_1$ such that

$$\mathbf{z}_0 \in P \times W \times \mathbb{R}^n, \quad \text{dist}(\mathbf{z}_0, \mathcal{A}_0) \leq d_2 \quad \Rightarrow \quad \text{dist}(\mathbf{z}(t, \mathbf{z}_0), \mathcal{A}_0) \leq M e^{-at} \text{dist}(\mathbf{z}_0, \mathcal{A}_0)$$

in which $\mathbf{z}(t, \mathbf{z}_0)$ denotes the solution of (5) passing through \mathbf{z}_0 at time $t = 0$. \triangleleft

As is well known, a fundamental step in the construction of a controller which solves the problem of output regulation is the design on an “internal model”, i.e. an autonomous dynamical system which generates all “feed-forward inputs capable to secure perfect tracking” (in the present case, the set of all inputs of the form $u(t) = c(\mathbf{z}(t))$, with $\mathbf{z}(t)$ a trajectory of the restriction of (5) to \mathcal{A}_0). In this respect, a relevant contribution to the design of internal models has been the methodology originally developed by Huang and co-authors (see e.g. [13, 11, 12]). As a matter of fact, in these works a precise characterization is provided of when the function $u(t)$ in question satisfies a linear differential equation

$$u^{(d)}(t) + a_{d-1}u^{(d-1)}(t) + \cdots + a_1u^{(1)}(t) + a_0u(t) = 0$$

which can then be taken as internal model and provides the fundamental core for the design of regulators. It was only recently, though, that attempts to weaken this condition have been taken. In [20], for instance, the case in which the coefficients of the above equation may depend on the uncertain (constant) parameter ϱ was considered. Alternative and relevant extensions have also been proposed in the recent works [7, 8]. The extension to the case in which the equation above is nonlinear (but ϱ -independent) was treated in [4].

The present paper proposes an extension, in the design of internal models, to the case in which the autonomous dynamical system that generates $u(t)$ is nonlinear but “linearizable by output injection”, with coefficients which possibly depend on the uncertain (constant) parameter ϱ . This case, to the best of our knowledge, has not been treated before by other authors. The assumption which characterizes when the extension in question is possible is the following one.

Assumption (iv): there exist a positive integer d , a C^1 map

$$\begin{aligned} \tau &: \mathcal{Z} \rightarrow \mathbb{R}^d \\ \mathbf{z} &\mapsto \tau(\mathbf{z}), \end{aligned}$$

a C^0 map

$$\begin{aligned} \theta &: P \rightarrow \mathbb{R}^q \\ \varrho &\mapsto \theta(\varrho), \end{aligned}$$

an observable pair $(A, C) \in \mathbb{R}^{d \times d} \times \mathbb{R}^{1 \times d}$, and two C^1 maps $\phi: \mathbb{R} \rightarrow \mathbb{R}^d$ and $\Omega: \mathbb{R} \rightarrow \mathbb{R}^{d \times q}$ such that the following identities (which we call *immersion property*)

$$\frac{\partial \tau}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z}) = A \tau(\mathbf{z}) + \phi(C \tau(\mathbf{z})) + \Omega(C \tau(\mathbf{z})) \theta(\varrho) \quad (6)$$

$$c(\mathbf{z}) = C \tau(\mathbf{z}) \quad (7)$$

hold for all $\mathbf{z} \in \mathcal{A}_0$, $\varrho \in P$. \triangleleft

Remark. Without loss of generality (see [17, page 208]), we can assume throughout that the matrices A and C in (8) have the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0 \quad \cdots \quad 0).$$

Furthermore, note that since the maps $\Omega(\cdot)$ and $\phi(\cdot)$ are continuously differentiable and the relations (6) – (7) are supposed to hold over the compact set \mathcal{A}_0 , it can be assumed without loss of generality that functions $\phi(\cdot)$ and $\Omega(\cdot)$ have compact support. This being the case, the functions in question can be assumed *globally Lipschitz*, i.e. there exist L_ϕ and L_Ω such that

$$|\phi(s_1) - \phi(s_2)| \leq L_\phi |s_1 - s_2|, \quad |\Omega(s_1) - \Omega(s_2)| \leq L_\Omega |s_1 - s_2|,$$

for all s_1, s_2 . \triangleleft

Assumption (iv) can be rephrased by saying that for each initial condition $\mathbf{z}(0) \in \mathcal{A}_0$ of (5), there is a pair $\xi(0), \theta$ such that the control input $u(t) = c(\mathbf{z}(\mathbf{t}))$ can be seen as output of a system of the form

$$\begin{aligned} \dot{\xi} &= A\xi + \phi(y) + \Omega(y)\theta \\ \dot{\theta} &= 0 \\ y &= C\xi. \end{aligned} \quad (8)$$

2.3 Comparison with earlier work on adaptive regulation

It may be useful to compare the assumption above with some of the standing assumptions in the earlier work [20] on adaptive regulation. In framework of [20], the lower subsystem of (2) was assumed to be a neutrally stable linear system

$$\dot{w} = S(\varrho)w.$$

Assumption III.1 of [20] was the existence of a map $\zeta : P \times W \rightarrow \mathbb{R}^n$ satisfying

$$\frac{\partial \zeta}{\partial w} S(\varrho)w = f_0(\varrho, w, \zeta(\varrho, w)),$$

or, what is the same, of a map $\zeta : P \times W \rightarrow \mathbb{R}^n$ whose graph is a compact invariant set for (4). Assumption V.1 of [20] was that the graph of ζ is globally asymptotically and locally exponentially stable. As a consequence, it is readily seen that Assumptions III.1 and V.1 of

[20] imply the fulfillment of assumptions (ii) and (iii) above and, in particular, the fact that the set \mathcal{A}_0 is the graph of the map ζ .

Assumption III.2 of [20] was the existence of an integer q , of a $q \times q$ matrix $\Phi(\varrho)$, continuously depending on ϱ , of a $1 \times q$ matrix Γ such that the pair $(\Phi(\varrho), \Gamma)$ is observable for all ϱ , and of a C^1 map $\tau : P \times W \rightarrow \mathbb{R}$ satisfying the pair of conditions

$$\begin{aligned} \frac{\partial \tau}{\partial w} S(\varrho) w &= \Phi(\varrho) \tau(\rho, w) \\ c(\varrho, w, \zeta(\varrho, w)) &= \Gamma \tau(\rho, w), \end{aligned}$$

which were referred to as conditions of “immersion” into a *linear* (observable) system. It is shown now that this assumption automatically yields assumption (iv) above.

Since the pair $(\Phi(\varrho), \Gamma)$ is observable for all ϱ , there exist a matrix $M(\varrho)$, continuously depending on ϱ , and continuous functions $\theta_1(\varrho), \dots, \theta_q(\varrho)$ such that

$$A(\varrho) := M(\varrho) \Phi(\varrho) M(\varrho)^{-1} = \begin{pmatrix} \theta_1(\varrho) & 1 & 0 & \dots & 0 \\ \theta_2(\varrho) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_q(\varrho) & 0 & 0 & \dots & 1 \end{pmatrix}$$

$$C := \Gamma M(\varrho)^{-1} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The matrix $A(\varrho)$ can be written in the form $A(\varrho) = A + \theta(\varrho)C$ in which $\theta(\varrho)$ is the column vector defined by the first column of $A(\varrho)$ and the pair (A, C) is observable. Setting $\bar{\tau}(\varrho, w) = M(\varrho) \tau(\varrho, w)$, and $\Omega(x) = xI_q$, it is readily seen that

$$\begin{aligned} \frac{\partial \bar{\tau}}{\partial w} S(\varrho) w &= A \bar{\tau}(\rho, w) + \Omega(C \bar{\tau}(\rho, w)) \theta(\varrho) \\ c(\varrho, w, \zeta(\varrho, w)) &= C \bar{\tau}(\rho, w). \end{aligned}$$

This, bearing in mind the fact that \mathcal{A}_0 is the graph of ζ , shows that assumption III.2 of [20] implies the fulfillment of assumption (iv) above.

The main limitation of the approach of [20] was the assumption of “immersion” into a *linear* system, namely the assumption that the set of inputs capable to secure perfect tracking is a subset of the set of solutions of a suitable *linear* (though ϱ -dependent) differential equation. To show that this limitation is overcome by the approach presented in this paper, we give here a simple but significant example.

Example. Consider the regulation problem

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\omega^2 w_1 \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\sigma z_1 - (z_1^2 - 1)z_2 - w_1 + e \\ \dot{e} &= -\mu z_1 + u \end{aligned}$$

in which ω , σ and μ are uncertain parameters ranging over compact sets. Note that ω is the uncertain frequency of a linear exosystem, while σ characterizes the frequency of a stable limit cycle embedded in the zero dynamics of the controlled plant. Note also that the equilibrium $z = 0$ of the zero dynamics is unstable.

Set $\varrho = \text{col}(\omega, \sigma, \mu)$ and $\mathbf{z} = \text{col}(\varrho, w, z)$. The function $c(\mathbf{z})$ in this case is the function $c(\mathbf{z}) = \mu z_1$. Set now

$$d(x, \mu) = \frac{1}{\mu^2} x^2 - 1$$

and

$$\begin{aligned}\tau_1(\mathbf{z}) &= \mu z_1 \\ \tau_2(\mathbf{z}) &= \mu z_2 + \int_0^{\mu z_1} d(x, \mu) dx \\ \tau_3(\mathbf{z}) &= \mu w_1 - \omega^2 \mu z_1 \\ \tau_4(\mathbf{z}) &= \mu w_2 - \omega^2 \left[\mu z_2 + \int_0^{\mu z_1} d(x, \mu) dx \right].\end{aligned}$$

A simple calculation shows that the map $\tau(\mathbf{z})$ thus defined satisfies conditions (6) and (7), with (A, C) an observable pair and

$$\phi_0(y) = \begin{pmatrix} y \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Omega(y) = \begin{pmatrix} -y^3 & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & -y \end{pmatrix}, \quad \theta(\varrho) = \begin{pmatrix} 1/\mu \\ \sigma \\ -\omega^2 + \sigma \\ -\omega^4 - \omega^2 \sigma \end{pmatrix}.$$

3 The adaptive internal model

3.1 The structure of the regulator

As pointed out in [4], the design of internal models can be reduced to the design of observers (*nonlinear* observers, in that case). In the presence of uncertain parameters in the exosystem, an *adaptive* observer should be used. Bearing in mind the construction of adaptive observers for systems which are linearizable by output injection pioneered by Bastin and Gevers [1], we consider in what follows a candidate controller of the form

$$\begin{aligned}u &= \xi_1 + v \\ \dot{\xi} &= A\xi + \phi(\xi_1) + \Omega(\xi_1)\hat{\theta} + H(X, \xi_1)v - M(X)dzv_\ell(\hat{\theta}) \\ \dot{\hat{\theta}} &= \beta(X, \xi_1)v - dzv_\ell(\hat{\theta}) \\ \dot{X} &= FX + G\Omega(\xi_1)\end{aligned} \tag{9}$$

in which ξ_1 denotes the first component of ξ , the matrix X is a $(d-1) \times q$ matrix, $M(X)$ is a $d \times q$ matrix defined as

$$M(X) = \begin{pmatrix} 0 \\ X \end{pmatrix},$$

while the vectors $H(X, \xi_1), \beta(X, \xi_1)$ and the matrices F, G have the form described below. The vector-valued *dead-zone* function $\text{dzv}_\ell(\cdot)$ is defined as

$$\text{dzv}_\ell(\text{col}(s_1, \dots, s_q)) = \text{col}(\text{dz}_\ell(s_1), \dots, \text{dz}_\ell(s_q))$$

in which $\text{dz}_\ell(\cdot)$ is any continuously differentiable function satisfying

$$\text{dz}_\ell(x) = \begin{cases} 0 & \text{if } |x| \leq \ell \\ x & \text{if } |x| \geq \ell + 1 \end{cases} \quad (10)$$

and the amplitude ℓ of the dead-zone is chosen so that

$$\ell > \max_{\varrho \in P} |\theta(\varrho)|.$$

This controller can be viewed as a “copy” of (8), corrected by an “innovation term”, augmented with an “adaptation law” for $\hat{\theta}$ and with a “filter” which generates the “auxiliary state” X . The additional input v , which is a “stabilizing control”, will eventually be taken as $v = -ke$.

The choice of the functions $H(X, \xi_1), \beta(X, \xi_1)$ and the matrices F, G of (9) is inspired by certain calculations used in [1] and [18] for the design of adaptive nonlinear observers. Define new variables

$$\begin{aligned} \tilde{\theta} &= \hat{\theta} - \theta(\varrho) \\ \eta &= \xi - M(X)\tilde{\theta}. \end{aligned} \quad (11)$$

(note that $\eta_1 = \xi_1$) and observe that, in the new variables, the second equation of (9) becomes

$$\begin{aligned} \dot{\eta} &= A(\eta + M(X)\tilde{\theta}) + \phi(\eta_1) + \Omega(\eta_1)(\theta(\varrho) + \tilde{\theta}) + H(X, \eta_1)v - M(\dot{X})\tilde{\theta} - M(X)\beta(X, \eta_1)v \\ &= A\eta + [AM(X) + \Omega(\eta_1) - M(\dot{X})]\tilde{\theta} + [H(X, \eta_1) - M(X)\beta(X, \eta_1)]v + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho). \end{aligned} \quad (12)$$

The third equation, instead, becomes

$$\dot{\tilde{\theta}} = \beta(X, \eta_1)v - \text{dzv}_\ell(\tilde{\theta} + \theta(\varrho)).$$

The choices of $H(X, \eta_1), \beta(X, \eta_1)$ and of F, G are meant to simplify the terms

$$[AM(X) + \Omega(\eta_1) - M(\dot{X})]\tilde{\theta} + [H(X, \eta_1) - M(X)\beta(X, \eta_1)]v$$

in the expression (12). First of all, note that choosing

$$H(X, \eta_1) = M(X)\beta(X, \eta_1) + K$$

with K a constant vector, to be fixed later, the second term becomes equal to Kv . The first term, instead, can be made equal to

$$[AM(X) + \Omega(\eta_1) - M(\dot{X})]\tilde{\theta} = b\beta^T(X, \eta_1)\tilde{\theta}$$

in which b is a $d \times 1$ fixed vector, if $M(X)$ satisfies

$$M(\dot{X}) = (A - bCA)M(X) + (I - bC)\Omega(\eta_1)$$

and $\beta(X, \eta_1)$ satisfies

$$\beta^T(X, \eta_1) = CAM(X) + C\Omega(\eta_1).$$

In this way, the second equation of (9) reduces to

$$\dot{\eta} = A\eta + b\beta^T(X, \eta_1)\tilde{\theta} + Kv + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho). \quad (13)$$

To show that the required differential equation for X can be enforced, pick a column vector $b = \text{col}(1, b_2, \dots, b_d)$. Then, bearing in mind the definition of $M(X)$, it is easily realized that the required differential equation holds if the matrices F and G have the form (see [18])

$$F = \begin{pmatrix} -b_2 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -b_{d-1} & 0 & \cdots & 0 & 1 \\ -b_d & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -b_2 & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ -b_{d-1} & 0 & \cdots & 0 & 1 & 0 \\ -b_d & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \quad (14)$$

In summary, the quantities $H(X, \xi_1), \beta(X, \xi_1), F, G$ which appear in the controller (9) are determined as follows: F and G are the matrices in (14), while $\beta(X, \xi_1)$ and $H(X, \xi_1)$ are chosen as

$$\beta(X, \xi_1) = [CA \begin{pmatrix} 0 \\ X \end{pmatrix} + C\Omega(\xi_1)]^T \quad (15)$$

$$H(X, \xi_1) = \begin{pmatrix} 0 \\ X \end{pmatrix} [CA \begin{pmatrix} 0 \\ X \end{pmatrix} + C\Omega(\xi_1)]^T + K. \quad (16)$$

The vectors b and K are left undetermined, for the time being.

The controller thus defined determines a closed loop system which, in the coordinates indicated above, can be written as

$$\begin{aligned} \dot{\varrho} &= 0 \\ \dot{w} &= s(\varrho, w) \\ \dot{z} &= f_0(\varrho, w, z) + f_1(\varrho, w, z, e)e \\ \dot{e} &= q(\varrho, w, z, e) + \eta_1 + v \\ \dot{\eta} &= A\eta + b\beta^T(X, \eta_1)\tilde{\theta} + Kv + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho) \\ \dot{\tilde{\theta}} &= \beta(X, \eta_1)v - dzv_\ell(\tilde{\theta} + \theta(\varrho)) \\ \dot{X} &= FX + G\Omega(\eta_1). \end{aligned} \quad (17)$$

This system, viewed as a system with input v and output e , has relative degree 1 and a zero dynamics characterized by the equations

$$\begin{aligned}
\dot{\varrho} &= 0 \\
\dot{w} &= s(\varrho, w) \\
\dot{z} &= f_0(\varrho, w, z) \\
\dot{\eta} &= A\eta - K[q(\varrho, w, z, 0) + \eta_1] + b\beta^T(X, \eta_1)\tilde{\theta} + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho) \\
\dot{\tilde{\theta}} &= -\beta(X, \eta_1)[q(\varrho, w, z, 0) + \eta_1] - \text{dzv}_\ell(\tilde{\theta} + \theta(\varrho)) \\
\dot{X} &= FX + G\Omega(\eta_1).
\end{aligned} \tag{18}$$

Standard arguments suggest that if the latter have convenient asymptotic properties, in particular possess a locally exponentially stable compact attractor, an additional control of the form $v = -ke$, (with large $k > 0$) should be able to keep trajectories bounded and steer $e(t)$ to zero. Thus, we proceed to analyze the properties of (18). The intuition that the expected asymptotic properties hold is based on the observation that, in view of Assumption (iv), the fourth equation of (18) can be regarded as the equation of an adaptive observer, constructed according to the methods of [1] and [18], for the variable $\tau(\mathbf{z})$, with the fifth equation providing of the appropriate adaptation law.

3.2 Trajectories of (18) are bounded

Let the initial conditions for ϱ, w, z be taken in the set \mathbf{Z} , a subset of a set \mathcal{Z} which by hypothesis is positively invariant for the (autonomous) subsystem consisting the top three equations of (18) and consider the change of variables

$$\chi = \eta - \tau(\varrho, w, z).$$

Using (6) and (7) (which, it should be borne in mind, hold only on \mathcal{A}_0) system (18) becomes

$$\begin{aligned}
\dot{\varrho} &= 0 \\
\dot{w} &= s(\varrho, w) \\
\dot{z} &= f_0(\varrho, w, z) \\
\dot{\chi} &= (A - KC)\chi + b\beta^T(X, \chi_1 + \tau_1)\tilde{\theta} + \Delta(\chi_1, \tau_1, \theta) + \varphi(\varrho, w, z) \\
\dot{\tilde{\theta}} &= -\beta(X, \chi_1 + \tau_1)\chi_1 - \text{dzv}_\ell(\tilde{\theta} + \theta(\varrho)) \\
\dot{X} &= FX + G\Omega(\chi_1 + \tau_1),
\end{aligned} \tag{19}$$

in which

$$\Delta(\chi_1, \tau_1, \theta) = \phi(\chi_1 + \tau_1) - \phi(\chi_1) + [\Omega(\chi_1 + \tau_1) - \Omega(\chi_1)]\theta(\varrho)$$

is a term which vanishes at $\chi_1 = 0$ and

$$\begin{aligned} \varphi(\varrho, w, z) = & K(c(\varrho, w, z) - \tau_1(\varrho, w, z)) + A\tau(\varrho, w, z) + \phi(\tau_1(\varrho, w, z)) + \Omega(\tau_1(\varrho, w, z))\theta(\varrho) \\ & - \frac{\partial \tau}{\partial z} f_0(\varrho, w, z) - \frac{\partial \tau}{\partial w} s(\varrho, w) \end{aligned} \quad (20)$$

is a term vanishing on \mathcal{A}_0 . In particular note that, since $\phi(\cdot)$ and $\Omega(\cdot)$ can be taken to be globally Lipschitz and θ ranges over a compact set, there exists a number L such that

$$|\Delta(\chi_1, \tau_1, \theta)| \leq L_\phi |\chi_1| + L_\Omega |\chi_1| |\theta| \leq L |\chi_1|$$

for all χ_1, τ_1, θ .

Mimicking a construction used in [18] for the design of adaptive observers, the vectors b_i 's and K are chosen as follows. The b_i 's be such that the polynomial

$$p(\lambda) = \lambda^{d-1} + b_2 \lambda^{d-2} + \dots + b_{d-1} \lambda + b_d \quad (21)$$

has $d - 1$ distinct roots with negative real part, in which case the matrix F in the bottom equation of (19) is Hurwitz (and has distinct eigenvalues). On the other hand, K is chosen as

$$K = Ab + \lambda b \quad (22)$$

in which $\lambda > 0$. These choices are such that the following result holds.

Lemma 1 *Suppose assumptions (i), (ii), (iv) hold. There is a number λ^* such that, if $\lambda \geq \lambda^*$, all trajectories of (19) are bounded.*

Proof. Because of assumption (ii), $(\varrho, w(t), z(t)) \in \mathcal{Z}$ for all $t \geq 0$, where \mathcal{Z} is a compact set. From this, standard arguments can be used to show that trajectories of (19) are defined for all $t \geq 0$ and, consequently, also $X(t)$ is bounded (recall that $|\Omega(\cdot)|$ has compact support). To prove the Lemma it remains to show that also $\chi(t)$ and $\tilde{\theta}(t)$ are bounded. To this end, let χ be partitioned as $\chi = \text{col}(\chi_1, \chi_2)$, in which χ_2 is a $(d - 1) \times 1$ vector and replace χ_2 by

$$\zeta = \hat{b}\chi_1 + \chi_2,$$

where $\hat{b} = -\text{col}(b_2, b_3, \dots, b_d)$. In this way, the fourth and fifth equations of (19) are changed to

$$\begin{aligned} \dot{\chi}_1 &= -\lambda\chi_1 + \hat{c}\zeta + \beta^T(X, \chi_1 + \tau_1)\tilde{\theta} + C\Delta(\chi_1, \tau_1, \theta) + C\varphi(\varrho, w, z) \\ \dot{\zeta} &= F\zeta + (\hat{b} \ I) \Delta(\chi_1, \tau_1, \theta) + (\hat{b} \ I) C\varphi(\varrho, w, z) \\ \dot{\tilde{\theta}} &= -\beta(X, \chi_1 + \tau_1)\chi_1 - \text{dzv}_\ell(\tilde{\theta} + \theta(\varrho)). \end{aligned} \quad (23)$$

Choose now for (23) the Lyapunov function

$$V(\chi_1, \zeta, \tilde{\theta}) = \chi_1^2 + \zeta^T P \zeta + \tilde{\theta}^T \tilde{\theta}, \quad (24)$$

in which P is the positive definite solution of $PF + F^T P = -I$ and obtain, after some simple algebra,

$$\begin{aligned} \dot{V} \leq & -2\lambda\chi_1^2 - |\zeta|^2 - 2\tilde{\theta}^T \text{d}z v_\ell(\tilde{\theta} + \theta(\varrho)) + L_1|\chi_1|^2 + L_2|\chi_1| |\zeta| \\ & + L_3|\chi_1| |\varphi(\varrho, w, z)| + L_4|\zeta| |\varphi(\varrho, w, z)| \end{aligned} \quad (25)$$

for some $L_i > 0$, $i = 1 \dots, 4$. Bearing in mind the fact that $\varphi(\varrho, w, z)$ is bounded by some fixed number $\bar{\varphi} > 0$ and completing the squares, one obtains

$$\dot{V} \leq -(2\lambda - L_1 + \frac{1}{2}L_2^2)\chi_1^2 - \frac{1}{2}|\zeta|^2 - 2\tilde{\theta}^T \text{d}z v_\ell(\tilde{\theta} + \theta(\varrho)) + L_3|\chi_1|\bar{\varphi} + L_4|\zeta|\bar{\varphi}. \quad (26)$$

A property of the function (10), in view of the choice of ℓ , is that

$$\tilde{\theta}^T \text{d}z v_\ell(\tilde{\theta} + \theta(\varrho)) \geq 0 \quad \text{for all } \tilde{\theta} \in \mathbb{R}^q \quad \text{and} \quad \varrho \in P. \quad (27)$$

Moreover, for any $\delta > \sqrt{q}(2\ell + 1)$ there is a number $c_1 > 0$ such that

$$|\tilde{\theta}| \geq \delta \quad \Rightarrow \quad 2\tilde{\theta}^T \text{d}z v_\ell(\tilde{\theta} + \theta(\varrho)) \geq c_1|\tilde{\theta}|^2 \quad \text{for all } \tilde{\theta} \in \mathbb{R}^q \quad \text{and} \quad \varrho \in P. \quad (28)$$

If λ is large enough, inequality (26), in view of property (28), yields

$$|\tilde{\theta}| \geq \delta \quad \Rightarrow \quad \dot{V} \leq -c_2|(\chi_1, \zeta, \tilde{\theta})|^2 + c_3|(\chi_1, \zeta, \tilde{\theta})|\bar{\varphi}$$

for suitable $c_2 > 0$, $c_3 > 0$. This, in turn, yields

$$|\tilde{\theta}| \geq \delta \quad \text{and} \quad |(\chi_1, \zeta, \tilde{\theta})| > \frac{c_3}{c_2} \bar{\varphi} \quad \Rightarrow \quad \dot{V} < 0. \quad (29)$$

Property (27), on the other hand, yields

$$\dot{V} \leq -c_2|(\chi_1, \zeta)|^2 + c_3|(\chi_1, \zeta)|\bar{\varphi}$$

which, in turn, yields

$$|(\chi_1, \zeta)| > \frac{c_3}{c_2} \bar{\varphi} \quad \Rightarrow \quad \dot{V} < 0. \quad (30)$$

At this point, it is easy to conclude that (29) and (30) imply

$$|(\chi_1, \zeta, \tilde{\theta})| > \sqrt{\delta^2 + \left(\frac{c_3}{c_2} \bar{\varphi}\right)^2} \quad \Rightarrow \quad \dot{V} < 0.$$

Bearing in mind the fact that $V(\chi_1, \zeta, \tilde{\theta})$ is a quadratic form, the claim follows by standard arguments. \triangleleft

3.3 The limit set of (18) and its properties

Let the initial conditions $\eta(0), \tilde{\theta}(0), X(0)$ of (18) be taken in fixed compact sets $\mathbf{H}, \Theta, \mathbf{X}$. From Lemma 1 we can claim that, if λ is large enough, the positive orbit of the set

$$\mathbf{B} = P \times W \times Z \times \mathbf{H} \times \Theta \times \mathbf{X}$$

under the flow of (18) is bounded. As a consequence $\omega(\mathbf{B})$, the ω -limit set of \mathbf{B} under the flow of (18), is a non-empty, compact and invariant set, which uniformly attracts all trajectories of (18) with initial conditions in \mathbf{B} . In what follows we render the structure of $\omega(\mathbf{B})$ explicit. To this end, we need an extra hypothesis, which – as any well-educated reader in adaptive control is expecting – plays in the present context a role completely similar to the role of the classical hypothesis of *persistence of excitation*.

Consider the equivalent system (19) and rewrite the three top equations as in (5) (consistently rewrite $\varphi(\varrho, w, z)$ as $\varphi(\mathbf{z})$ and $\tau(z, w, \varrho)$ as $\tau(\mathbf{z})$). Because of the special triangular structure of (19), it can be observed that if $(\mathbf{z}, \chi, \tilde{\theta}, X)$ is a point of $\omega(\mathbf{B})$, necessarily \mathbf{z} is a point in the ω -limit set of \mathbf{Z} under the flow of (5), that is, \mathbf{z} is a point of \mathcal{A}_0 . This implies that on $\omega(\mathbf{B})$ we have $\varphi(\mathbf{z}) = 0$ and thus system (19) simplifies as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\chi} &= (A - KC)\chi + b\beta^T(X, \chi_1 + \tau_1)\tilde{\theta} + \Delta(\chi_1, \tau_1, \theta) \\ \dot{\tilde{\theta}} &= -\beta(X, \chi_1 + \tau_1)\chi_1 - \text{d}z\mathbf{v}_\ell(\tilde{\theta} + \theta(\varrho)) \\ \dot{X} &= FX + G\Omega(\chi_1 + \tau_1). \end{aligned} \tag{31}$$

It will be shown now that, if the announced additional hypothesis of persistency of excitation holds, points of $\omega(\mathbf{B})$ have necessarily $\chi = 0, \tilde{\theta} = 0$. To introduce the hypothesis in question we observe first of all the following interesting feature.

Lemma 2 *The graph of the map*

$$\begin{aligned} \sigma : \mathcal{A}_0 &\rightarrow \mathbb{R}^{(d-1) \times q} \\ \mathbf{z} &\mapsto X = \int_{-\infty}^0 e^{-Fs} G\Omega(\tau_1(\mathbf{z}(s, \mathbf{z})))ds \end{aligned}$$

is invariant for

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{X} &= FX + G\Omega(\tau_1(\mathbf{z})). \end{aligned} \tag{32}$$

Proof. Let $\mathbf{z}(t, \mathbf{z}_0)$ denote the solution of (5) passing through \mathbf{z}_0 at time $t = 0$ and note that, if $\mathbf{z}_0 \in \mathcal{A}_0$, then $\mathbf{z}(t, \mathbf{z}_0) \in \mathcal{A}_0$ for all t . Since F is a Hurwitz matrix, the map $\sigma(\cdot)$ is well defined. A simple calculation shows that

$$\sigma(\mathbf{z}(t, \mathbf{z}_0)) = e^{Ft}\sigma(\mathbf{z}_0) + \int_0^t e^{F(t-s)}G\Omega(\tau_1(\mathbf{z}(s, \mathbf{z}_0)))ds,$$

from which the result follows. \triangleleft

The new assumption is the following one.

Assumption (v): Consider the map $\gamma : \mathcal{A}_0 \rightarrow \mathbb{R}^{q \times 1}$ defined as

$$\gamma : \mathbf{z} \mapsto \beta(\sigma(\mathbf{z}), \tau_1(\mathbf{z}))$$

It is assumed that for any initial condition $\mathbf{z}_0 \in \mathcal{A}_0$ the identity

$$c^T \gamma(\mathbf{z}(t, \mathbf{z}_0)) = 0, \quad \text{for all } t \in \mathbb{R}$$

implies $c = 0$. \triangleleft

Under this hypothesis, the set $\omega(\mathbf{B})$ assumes a very simple structure. As a matter of fact, the following result holds.

Proposition 1 *Under the assumptions (i), (ii), (iv) and (v) the set $\omega(\mathbf{B})$ is the graph of a continuous map defined on \mathcal{A}_0 . Any point of $\omega(\mathbf{B})$ is a point $(\mathbf{z}, \eta, \tilde{\theta}, X)$ in which $\mathbf{z} \in \mathcal{A}_0$ and*

$$\eta = \tau(\mathbf{z}), \quad \tilde{\theta} = 0, \quad X = \sigma(\mathbf{z}).$$

If also assumption (iii) holds, then $\omega(\mathbf{B})$ is locally exponentially attractive for (18).

Proof. By contradiction, suppose a point $\mathbf{p} = (\mathbf{z}, \chi_0, \tilde{\theta}_0, X)$ with either $\chi_0 \neq 0$ or $\tilde{\theta}_0 \neq 0$ is in $\omega(\mathbf{B})$. Since $\omega(\mathbf{B})$ is compact and invariant, the backward trajectory of (31) starting at this point is bounded. Along this trajectory, the Lyapunov function (24) satisfies $V \leq C$ for all $t \leq 0$, for some $C > 0$. Moreover, since $\varphi(\mathbf{z}) = 0$ on $\omega(\mathbf{B})$, the same computations indicated in the proof of Lemma 1 show that

$$\dot{V}(t) \leq -(2\lambda - L_1 - \frac{1}{2}L_2^2)|\chi_1(t)|^2 - \frac{1}{2}|\zeta(t)|^2 - 2\tilde{\theta}(t)\text{d}z\text{v}_\ell(\tilde{\theta}(t) + \theta(\varrho)).$$

Using property (27) it is seen that, if $\lambda \geq \lambda^*$, $V(t)$ is non-increasing along trajectories. As consequence, since $V(t)$ is bounded, there exists a finite number V_α such that $\lim_{t \rightarrow -\infty} V(t) = V_\alpha$. The trajectory in question is attracted, in backward time, by its own α -limit set $\alpha(\mathbf{p})$, which, as it is well known, is nonempty, compact and invariant. Moreover, by definition, the function $V(\chi_1, \zeta, \tilde{\theta})$ has the same value V_α at any point of $\alpha(\mathbf{p})$. Proceeding as in the classical proof of LaSalle's invariance principle, pick an initial condition $\hat{\mathbf{p}}$ in the set $\alpha(\mathbf{p})$ and consider the corresponding trajectory of (31). Along such trajectory, $V(t)$ is constantly equal to V_α and hence

$$\chi_1(t) = 0, \quad \zeta(t) = 0, \quad \text{d}z\text{v}_\ell(\tilde{\theta}(t) + \theta(\varrho)) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Entering these constraints in (31), and observing that the vector b is nonzero, it is seen that necessarily

$$\begin{aligned}\tilde{\theta}^T \beta &= 0 \\ \dot{\tilde{\theta}} &= 0 \\ \dot{X} &= FX + G\Omega(\tau_1(\mathbf{z})).\end{aligned}$$

It is seen from this that $\tilde{\theta}(t)$ is a constant, say $\tilde{\theta}^*$, along such trajectory, while $X(t)$ is a solution of (32). Since F is Hurwitz and has distinct eigenvalues, $X(t)$ can be bounded for $t \leq 0$ only if $X(0) = \sigma(\mathbf{z}(0))$, where $\sigma(\cdot)$ is the map introduced in Lemma 2, in which case $X(t) = \sigma(\mathbf{z}(t))$. Since $X(t)$ has to be bounded because $\alpha(\mathbf{p})$ is compact, it follows that $X(t)$ is necessarily equal to $\sigma(\mathbf{z}(t))$. This being the case, bearing in mind the expression of $\beta(\cdot, \cdot)$ and the definition of the map $\gamma(\cdot)$, the first condition shows that necessarily

$$(\tilde{\theta}^*)^T \gamma(\mathbf{z}(t)) = 0, \quad \text{for all } t \in \mathbb{R}.$$

Thus, from Assumption (v), it is concluded that $\tilde{\theta}^* = 0$. As a consequence $(\chi_1, \zeta, \tilde{\theta}) = (0, 0, 0)$ at any point of $\alpha(\mathbf{p})$, and $V_\alpha = 0$. But this is a contradiction, because $V(t)$ is non-increasing along trajectories and $V(0)$ is strictly positive, if either $\chi_0 \neq 0$ or $\tilde{\theta}_0 \neq 0$.

The proof that $\omega(\mathbf{B})$ is locally exponentially stable is a consequence of Assumption (iii). It requires appropriate modifications of arguments used in similar instances in the proof of convergence of parameter estimates in various adaptive control schemes, such as those presented in various Chapters of [16], and is not included here for reasons of space. Details can be found in [9]. \triangleleft

4 Adaptive output regulation

We return now to the closed loop system obtained from the interconnection of (1), (2) and (9). As observed, this system, viewed as a system with input v and output e has relative degree 1. To put it in “normal form” we use, instead of (11), the change of variables

$$\begin{aligned}\tilde{\theta} &= \hat{\theta} - \theta(\rho) - \int_0^e \beta(X, \xi_1 - CKe + CKs) ds \\ \eta &= \xi - M[\hat{\theta} - \theta(\rho)] - Ke.\end{aligned}\tag{33}$$

This, after some simple algebra and some obvious rearrangement of terms, yields a system

of the form

$$\begin{aligned}
\dot{\varrho} &= 0 \\
\dot{w} &= s(\varrho, w) \\
\dot{z} &= f_0(\varrho, w, z) + f_1(\varrho, w, z, e)e \\
\dot{\eta} &= A\eta + b\beta^T\tilde{\theta}(\varrho) - K[q(\varrho, w, z, 0) + \eta_1] + \phi(\eta_1) + \Omega(\eta_1)\theta + \delta_1(\varrho, w, z, e, X, \eta_1) e \\
\dot{\tilde{\theta}} &= -\beta[q(\varrho, w, z, 0) + \eta_1] + \delta_2(\varrho, w, z, e, X, \eta_1) e - \text{d}z\text{v}_\ell(\tilde{\theta} + \theta(\varrho)) \\
\dot{X} &= FX + G\Omega(\eta_1) + \delta_3(\eta_1, e) e \\
\dot{e} &= -[q(\varrho, w, z, 0) + \eta_1] + \vartheta(\varrho, w, z, e)e + v.
\end{aligned} \tag{34}$$

in which $\delta_1(\cdot)$, $\delta_2(\cdot)$, $\delta_3(\cdot)$ and $\vartheta(\cdot)$ are continuously differentiable functions of their arguments. The notation β stands for $\beta(X, \eta_1 + CKe)$.

This system can be put in a much more compact form by setting

$$\mathbf{x} = \text{col}(\varrho, w, z, \eta, \tilde{\theta}, X_1, \dots, X_q),$$

(where X_i denotes the i -th column of X) which yields a system of the form

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}_0(\mathbf{x}) + \mathbf{F}_1(\mathbf{x}, e)e \\
\dot{e} &= \mathbf{h}(\mathbf{x}) + \mathbf{k}(\mathbf{x}, e)e + v.
\end{aligned} \tag{35}$$

In this notation, the “subsystem” $\dot{\mathbf{x}} = \mathbf{F}_0(\mathbf{x})$ is precisely system (18), while the function $\mathbf{h}(\mathbf{x})$, which is precisely the quantity $-[q(\varrho, w, z, 0) + \eta_1]$ in (34), vanishes on the set $\omega(\mathbf{B})$. Having realized this, it is not difficult to claim that the controller (9) completed with

$$v = -ke, \tag{36}$$

if k is sufficiently large, keeps trajectories bounded and steers $e(t)$ to 0.

Proposition 2 *Consider system (1) with exosystem (2). Let \mathbf{Z}, E be fixed compact sets of initial conditions, for which the assumptions (i)-(iv) indicated in section 2 are supposed to hold. Suppose, in addition, that assumption (v) introduced in section 3.3 holds. Consider the controller (9) completed with (36) and initial conditions in a fixed compact set \mathbf{K} . Then, there exists a number $k^* > 0$ such that if $k \geq k^*$ the positive orbit of $\mathbf{Z} \times E \times \mathbf{K}$ in the closed loop system is bounded and $e(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. The closed-loop system

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}_0(\mathbf{x}) + \mathbf{F}_1(\mathbf{x}, e)e \\
\dot{e} &= \mathbf{h}(\mathbf{x}) + \mathbf{k}(\mathbf{x}, e)e - ke.
\end{aligned} \tag{37}$$

can be viewed as interconnection of two subsystems, one with state \mathbf{x} and input e , the other with state e and input \mathbf{x} . As shown in [5], the upper subsystem is input-to-state stable (relative to the set $\omega(B)$), with a gain function which is linear if the assumptions (i) through (v) hold. Then, an easy extension (to the case of systems which are input-to-state stable relative to compact attractors) of the small-gain theorem of [15] can be invoked to show that, if k is large enough, the results of the Proposition hold. Further details on the proof of this “high-gain” stabilizability property can be found in [5]. \triangleleft

Before concluding the paper, we wish to point out the fact that the theory developed so far lends itself to deal with more general situations in which the equations of the (augmented) *exosystem* are *affected by regulated variable e* . Specifically, suppose that ϱ and w , instead of (2), are generated by a system of the more general form

$$\begin{aligned}\dot{\varrho} &= s_{\varrho}(\varrho, w, z, e)e \\ \dot{w} &= s(\varrho, w) + s_w(\varrho, w, z, e)e.\end{aligned}\tag{38}$$

In this case, in fact, the closed loop system obtained from the interconnection of (1), (38) and (9) can still be put in the form (35), in which $\mathbf{f}_0(\mathbf{z})$ has exactly the form (5), while

$$\mathbf{f}_1(\mathbf{z}, e) = \begin{pmatrix} s_{\varrho}(\varrho, w, z, e) \\ s_w(\varrho, w, z, e) \\ f_1(\varrho, w, z, e) \end{pmatrix}.$$

Thus, the result expressed by Proposition 2 continues to hold. Models of this kind occur, for instance, in the problem of self-compensation of mechanical and/or electrical asymmetries in a rotating electrical drives (see [2]).

5 Conclusions

This paper has discussed the design of nonlinear internal models in the problem of adaptive output regulation. It has been shown how the theory of adaptive observers can be successfully used to deal with complex situations, not covered by existing results, in which the desired steady state control input is generated by a possibly nonlinear system and depends on constant uncertain parameters. The result is framed in the general “non-equilibrium” theory proposed in [3], thus allowing for controlled plant with not necessarily stable zero dynamics.

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